Angular Momentum

Note Title

If the system's potential energy is istropic (i.e. $V(\vec{r}) = U(r)$ independent of the direction), energy and angular momentum are both conserved in classical mechanics. The angular momentum plays an even more important role in quantum mechanics.

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\Rightarrow \quad L_x = y P_z - z P_y$$
$$L_y = z P_x - x P_z$$
$$L_z = x P_y - y P_x$$

$$[L_x, L_y] = [y P_z - z P_y, z P_x - x P_z]$$
$$= [y P_z, z P_x] - [z P_y, z P_x]$$
$$- [y P_z, x P_z] + [z P_y, x P_z]$$
$$= y P_x [P_z, z] - 0 - 0 + P_y x [z, P_z]$$
$$= i\hbar (x P_y - y P_x) = i\hbar L_z$$

$$[L_y, L_z] = i\hbar L_x, \quad [L_z, L_x] = i\hbar L_y$$

$$\sigma_{L_x}^2 \sigma_{L_y}^2 \geq \left(\frac{1}{2i} \langle [L_x, L_y] \rangle \right)^2$$
$$= \left(\frac{\hbar}{2} \langle L_z \rangle \right)^2$$

$$\Rightarrow \sigma_{L_x} \sigma_{L_y} \geq \frac{\hbar}{2} |\langle L_z \rangle|$$

This implies that L_x and L_y cannot be simultaneously measured.

In other words, L_x and L_y cannot share the same eigenfunctions.

However L^2 and L_x (or L_y or L_z) can share the same eigenfunctions, because $[L^2, L_x] = 0 = [L^2, L_y]$
$= [L^2, L_z]$

So although we cannot measure L_x, L_y, L_z simultaneously, we can still measure L^2 and one of $\vec{L}$ components simultaneously. We can call that component just L_z.

So our objective is to find the simultaneous eigenstates of L^2 and L_z such that

$$L^2 f = \lambda f \quad \text{and} \quad L_z f = \mu f$$

and their eigenvalues.

We will look for the eigenvalues first, and eigenftns later.

For the eigenvalues, it is easiest to use the operator method as we did with the harmonic oscillator.

Let's define $L_\pm \equiv L_x \pm i L_y$

[Recall $a_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x)$]

$[L_z, L_\pm] = [L_z, L_x \pm i L_y]$

$= [L_z, L_x] \pm i [L_z, L_y]$

$= i\hbar L_y \mp i(i\hbar L_x)$

$= i\hbar(L_y \mp i L_x) = \mp i \cdot i\hbar (L_x \pm i L_y)$

$= \pm\hbar L_\pm$

And $[L^2, L_\pm] = 0$ $\because [L^2, L_i] = 0$

If f is an eigen-ftn of L^2 and L_z, that is $L^2 f = \lambda f$ and $L_z f = \mu f$, so is also $L_\pm f$, because

$L^2(L_\pm f) = L_\pm(L^2 f) = L_\pm(\lambda f) = \lambda(L_\pm f)$

$$L_z(L_\pm f) = (L_\pm L_z \pm \hbar L_z) f$$
$$= L_\pm (\mu \pm \hbar) f = (\mu \pm \hbar)(L_\pm f).$$

So $L_\pm f$ is a new eigenstate of L_z with the new eigenvalue $\mu \pm \hbar$ and is also an eigenstate of L^2 with the same eigenvalue of λ.

If we keep applying L_+, then we will eventually reach a "top rung" such that

$$L_+ f_t = 0,$$

because the maximum eigenvalue of L_z is limited by the fixed value of "λ", which is the eigenvalue of L^2

If we call the maximum eigenvalue of L_z by $\hbar l$,

$$L_z f_t = \hbar l f_t, \quad L^2 f_t = \lambda f_t$$

Now Let's see if we can find λ out of $\hbar l$.

$$L_\pm L_\mp = (L_x \pm i L_y)(L_x \mp i L_y)$$
$$= L_x^2 + L_y^2 \mp i(L_x L_y - L_y L_x)$$

$$= L_x^2 + L_y^2 \mp i\,(i\hbar L_z)$$
$$= L_x^2 + L_y^2 \pm \hbar L_z = L^2 - L_z^2 \pm \hbar L_z$$

$$\Rightarrow L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z$$

$$\Rightarrow L^2 f_t = (L_- L_+ + L_z^2 + \hbar L_z) f_t$$
$$= (L_z^2 + \hbar L_z) f_t$$
$$= (\hbar^2 \ell^2 + \hbar^2 \ell) f_t$$
$$= \hbar^2 \ell(\ell+1) f_t$$

$$\therefore \quad \lambda = \hbar^2 \ell(\ell+1)$$

In a similar way, there must be a bottom rung f_b, such that

$$L_- f_b = 0$$

Then if we call the eigenvalue of L_z for f_b by $\hbar\bar{\ell}$,

$$L_z f_b = \hbar\bar{\ell} f_b, \quad L^2 f_b = \lambda f_b$$

$$\Rightarrow L^2 f_- = (L_+ L_- + L_z^2 - \hbar L_z) f_-$$
$$= (L_z^2 - \hbar L_z) f_- = (\hbar^2 \bar{\ell}^2 - \hbar^2 \bar{\ell}) f_-$$

$$\Rightarrow \quad \lambda = \hbar^2 \bar{\ell}(\bar{\ell} - 1)$$

Comparing with the above,

$$l(l+1) = \bar{l}(\bar{l}-1)$$

$$\Rightarrow l^2 - \bar{l}^2 + l + \bar{l} = 0$$

$$\Rightarrow (l+\bar{l})(l-\bar{l}+1) = 0$$

$\Rightarrow \bar{l} = -l$ or $l+1$, but since $\bar{l} < l$

, $\underline{\bar{l} = -l}$

So the spectrum of eigenvalues for L_z must be

$$-\hbar l, -\hbar l+\hbar, \cdots \hbar l-\hbar, \hbar l$$

In other words, they should be $m\hbar$ with m being

$$-l, -l+1, \cdots l-1, l$$

So $l = -l + N$, with some integer N.

$$\Rightarrow l = \frac{N}{2}.$$

Thus possible values of l are

$$l = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$$

* Note here that the algebraic method allows not only integers but also half integers.

Summarizing this,

$$L^2 f_l^m = \hbar^2 l(l+1) f_l^m, \quad L_z f_l^m = \hbar m f_l^m$$

with $m = -l, -l+1, \cdots l-1, l$

Here note that, maximum eigenvalue of L_z, $\hbar l$, is smaller than maximum $\sqrt{\hbar^2 l(l+1)} = \hbar\sqrt{l(l+1)}$.

We CANNOT draw $\vec{L}$ by an arrow as we do in classical mechanics, because we cannot simultaneously know L_x, L_y and L_z.

Eigenfcns

Eigenfcns of L_z and L^2 cannot be obtained by the algebraic method, and we need to use the derivative forms of the operators and solve the differential equation.

From $\vec{L} = \vec{r} \times \vec{p} = \frac{\hbar}{i} \vec{r} \times \nabla$

with $\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$

$$\vec{L} = \frac{\hbar}{i}\left(\hat{\phi}\frac{\partial}{\partial\theta} - \hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$\Rightarrow \underline{L_z = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}}$$

and from $L^2 = L_+L_- + L_z^2 - \hbar L_z$

$$\Rightarrow L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$$

As we discussed in the first class of 3D Schrödinger equation earlier,

$$L_z f_\ell^m = \hbar m f_\ell^m$$

$$L^2 f_\ell^m = \hbar^2 \ell(\ell+1) f_\ell^m$$

$\Rightarrow f_\ell^m = Y_\ell^m(\theta, \varphi)$, the spherical Harmonics if we solve the differential equations.

Remember that the T.I.S.E. was

$$\frac{-\hbar^2}{2m}\cdot\frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{L^2}{\hbar^2}\right)\psi + U\psi = E\psi$$